A MAXIMAL INEQUALITY FOR THE TAIL OF THE BILINEAR HARDY-LITTLEWOOD FUNCTION

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Abstract. Let $(X, \mathcal{B}, \mu, T)$ be an ergodic dynamical system on a non-atomic finite measure space. We assume without loss of generality that $\mu(X) = 1$. Consider the maximal function $R^* : (f, g) \in L^p \times L^q \to R^*(f, g)(x) = \sup_{n \geq 1} \dfrac{f(T^n x) g(T^{2n} x)}{n}$. We obtain the following maximal inequality. For each $1 < p \leq \infty$ there exists a finite constant C_p such that for each $\lambda > 0$, and nonnegative functions $f \in L^p$ and $g \in L^1$

$$\mu\{x : R^*(f, g)(x) > \lambda\} \leq C_p \left(\frac{\|f\|_p \|g\|_1}{\lambda} \right)^{1/2}.$$

We also show that for each $\alpha > 2$ the maximal function $R^*(f, g)$ is a.e. finite for pairs of functions $(f, g) \in (L(\log L)^{2\alpha}, L^1)$.

1. Introduction

Let $(X, \mathcal{B}, \mu, T)$ be an ergodic dynamical system on a non-atomic finite measure space. We assume without loss of generality that $\mu(X) = 1$.

In [1] we proved the following maximal inequality about the maximal function $R^*(f, g)(x) = \sup_{n \geq 1} \dfrac{f(T^n x) g(T^{2n} x)}{n}$. For each $1 < p \leq \infty$, there exists a finite constant C'_p such that for each $\lambda > 0$, for every $f \in L^p, f > 1$ and $g \in L^1, g > 1$

$$\mu\{x : R^*(f, g)(x) > \lambda\} \leq C'_p \left(\frac{\|f\|_p^p \|g\|_1}{\lambda} \right)^{1/2}. \tag{1}$$

Furthermore the constant C'_p behaves like $\frac{1}{p-1}$ when p tends to 1. To be more precise, we will use that there exists $\widetilde{C}'$ such that for any $1 < p < 2$ we have

$$C'_p \leq \frac{\widetilde{C}'}{p-1}. \tag{2}$$

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Inequality (1) was enough to prove the a.e. convergence to zero of the tail $\dfrac{f(T^n x)g(T^{2n}x)}{n}$ of the double recurrence averages $\dfrac{1}{n}\sum_{k=1}^{n} f(T^k x)g(T^{2k}x)$ for pairs of functions (f,g) in $L^p \times L^1$ (or $L^1 \times L^p$) as soon as $p > 1$. On the other hand, in [2] the tail is used to show that these averages do not converge a.e. for pairs of (L^1, L^1) functions.

During the 2007 Ergodic Theory workshop at UNC-Chapel Hill, J.P. Conze asked if this inequality could be made homogeneous with respect to f and g. In this paper first we derive from (1) the following homogeneous version.

Theorem 1. *For each $1 < p < \infty$ there exists a finite constant C_p such that for each $f, g \geq 0$ and for all $\lambda > 0$ we have*

$$\mu\Big\{x : \sup_{n\geq 1} \frac{f(T^n x)g(T^{2n}x)}{n} > \lambda\Big\} \leq C_p \left(\frac{\|f\|_p \|g\|_1}{\lambda}\right)^{1/2}, \tag{3}$$

and there exists $\widetilde{C}$ such that for any $1 < p < 2$ we have

$$C_p \leq \frac{\widetilde{C}}{p-1}. \tag{4}$$

At the same meeting a question was raised about the a.e. finiteness of $R^*(f,g)$ for pairs of functions in $(L \log L, L^1)$. Our second result is based on an adaptation of Zygmund's extrapolation method [4] (vol. II, ch. XII, pp. 119-120) to $R^*(f,g)$. With somewhat crude estimates we prove the following theorem.

Theorem 2. *If $\alpha > 2$ and the pair of nonnegative functions (f,g) belongs to $(L(\log L)^{2\alpha}, L^1)$ then $R^*(f,g) = \sup_{n\geq 1} \frac{f(T^n x)g(T^{2n}x)}{n}$ is a.e. finite.*

2. Proofs

Proof of Theorem 1. First we can notice that the original inequality (1) is homogeneous with respect to the L^1 function g. Indeed, a simple change of variables shows that the case $g > t$ can easily be obtained from the case $g > 1$ with the same constant C'_p. So by approximating g with $g_n(x) = \max\{g(x), 1/n\}$ we can see that (1) holds if the assumption $g > 1$ is replaced by $g \geq 0$. Without loss of generality we can also suppose in the sequel that $\|g\|_1 = 1$.

If $\|f\|_p = 0$ we have nothing to prove. Otherwise, if we can show that (3) holds for $\widetilde{f} = f/\|f\|_p$ for all $\lambda > 0$, then this implies that it is true for f as well for all $\lambda > 0$. Thus, we just need to prove (3) for $f \in L^p$ with $\|f\|_p = 1$.

Set

$$M = \mu\Big\{x : \sup_{n\geq 1} \frac{f(T^n x)g(T^{2n}x)}{n} > \lambda\Big\}$$

and $h = \max\{f, 1\}$. By our remark about the assumption $g \geq 0$ the maximal inequality (1) is applicable and we obtain that $M \leq C_p'\big(\frac{\|h\|_p^p}{\lambda}\big)^{1/2}$, and (2) also holds for $1 < p < 2$. As $\|h\|_p \leq \|\mathbf{1}\|_p + \|f\|_p = 2$ we have the estimate

$$M \leq 2^{p/2} C_p' \left(\frac{1}{\lambda}\right)^{1/2} = 2^{p/2} C_p' \left(\frac{\|f\|_p \|g\|_1}{\lambda}\right)^{1/2},$$

with C_p' satisfying (2) for $1 < p < 2$. Therefore, we obtain

$$\mu\Big\{x : \sup_n \frac{f(T^n x) g(T^{2n} x)}{n} > \lambda\Big\} \leq 2^{p/2} C_p' \left(\frac{\|f\|_p \|g\|_1}{\lambda}\right)^{1/2} \leq C_p \left(\frac{\|f\|_p \|g\|_1}{\lambda}\right)^{1/2}$$

with $C_p = 2^{p/2} C_p'$ and from (2) it follows that there exists $\widetilde{C}$ such that (4) holds for $1 < p < 2$. $\square$

Proof of Theorem 2. The starting point is (3) and (4).

There exists a finite constant $\widetilde{C}$ such that for every $1 < p < 2$, for each $f, g \geq 0$ and for all $\lambda > 0$ we have

$$\mu\Big\{x : \sup_n \frac{f(T^n x) g(T^{2n} x)}{n} > \lambda\Big\} \leq \frac{\widetilde{C}}{p-1} \left(\frac{\|f\|_p \|g\|_1}{\lambda}\right)^{1/2}. \tag{5}$$

We can again assume without loss of generality that $\|g\|_1 = 1$. We fix the function g and denote by $R^*(f)(x)$ the maximal function $\sup_n \frac{f(T^n x) g(T^{2n} x)}{n}$. Now we can rewrite (5) as

$$\mu\big\{x : R^*(f)(x) > \lambda\big\} \leq \frac{\widetilde{C}}{p-1} \left(\frac{\|f\|_p}{\lambda}\right)^{1/2}. \tag{6}$$

The important element for the extrapolation is the factor $\frac{1}{p-1}$ in the above inequality.

Our goal is to prove that for $\alpha > 2$ there is C_α such that for any $f \in L(\log L)^{2\alpha}$ we have for each $\lambda > 0$

$$\mu\big\{x : R^*(f)(x) > \lambda\big\} \leq C_\alpha \frac{1 + \left(\int |f| (\log^+ |f|)^{2\alpha}\right)^{1/2}}{\lambda^{1/2}}. \tag{7}$$

Let γ_j be a positive sequence of numbers such that $\sum_{j=0}^{\infty} \gamma_j = 1$.

The function f being in $L(\log L)^{2\alpha}$ we have $\sum_{j=1}^{\infty} j^{2\alpha} 2^j \mu\big\{x : 2^j \leq f < 2^{j+1}\big\} < \infty$. We denote by t_j the quantity $\mu\big\{2^j \leq f < 2^{j+1}\big\}$, by f_j the function $2^j \mathbf{1}_{\{x : 2^j \leq f < 2^{j+1}\}}$

and by p_j the number $1+\frac{1}{j}$. We set $f_0(x)=f(x)$ if $0\le f(x)<2$, otherwise we put $f_0(x)=0$. Then

$$f\le 2\sum_{j=0}^{\infty} f_j. \tag{8}$$

We also have

$$\mu\{x: R^*(f_0)(x)>\frac{\lambda\gamma_0}{2}\}\le \tag{9}$$

$$\mu\{x: R^*(2\cdot \mathbf{1}_X)(x)>\frac{\lambda\gamma_0}{2}\}\le \frac{4\|g\|_1}{\lambda\gamma_0}=\frac{4}{\lambda\gamma_0}$$

by the standard maximal inequality for the ergodic averages (see [3] for instance).

For $j\ge 1$ by (6) used with $p_j=1+\frac{1}{j}$ we obtain

$$\mu\{x: R^*(f_j)(x)>\frac{\lambda\gamma_j}{2}\}\le \tag{10}$$

$$\widetilde{C}\frac{1}{(1+(1/j))-1}\left(\frac{2^{j/2}[t_j]^{1/2p_j}}{(\lambda\gamma_j/2)^{1/2}}\right)\le \sqrt{2}\widetilde{C}\frac{j2^{j/2}[t_j]^{1/2p_j}}{(\lambda\gamma_j)^{1/2}}.$$

We choose $\gamma_0=1/2$ and $\gamma_j=\frac{C_\gamma}{j(\log(j+1))^\beta}$ with $\beta>1$ and C_γ such that $\sum_{j=0}^{\infty}\gamma_j=1$.

Set $\widehat{C}=\dfrac{\sqrt{2}\widetilde{C}}{C_\gamma^{1/2}}$.

Using (8) and adding (9) and (10) for all j we obtain

$$\mu\{x: R^*(f)(x)>\lambda\}\le \sum_{j=0}^{\infty}\mu\{R^*(f_j)>\frac{\lambda\gamma_j}{2}\}\le \frac{8}{\lambda}+\sqrt{2}\widetilde{C}\sum_{j=1}^{\infty}\frac{j2^{j/2}\big[t_j\big]^{1/2p_j}}{(\lambda\gamma_j)^{1/2}}\le \tag{11}$$

$$\frac{8}{\lambda}+\widehat{C}\frac{\sum_{j=1}^{\infty} j^{3/2}[\log(j+1)]^{\beta/2}2^{j/2}\big[t_j\big]^{1/2p_j}}{\lambda^{1/2}}=\frac{8}{\lambda}+\widehat{C}\frac{A_1}{\lambda^{1/2}}.$$

To estimate A_1 denote by J_1 the set of those j for which $t_j^{1/2p_j}\le 3^{-j}$. Then

$$\sum_{j\in J_1} j^{3/2}[\log(j+1)]^{\beta/2}2^{j/2}\big[t_j\big]^{1/2p_j}\le \sum_{j=1}^{\infty} j^{3/2}[\log(j+1)]^{\beta/2}2^{j/2}3^{-j}\stackrel{\text{def}}{=}C_s. \tag{12}$$

If $j\notin J_1$ then $t_j^{1/2p_j}>3^{-j}$, that is,

$$3>t_j^{\frac{-\frac{1}{j}}{2p_j}}=t_j^{\frac{1-(1+\frac{1}{j})}{2p_j}}=t_j^{\frac{1}{2p_j}-\frac{1}{2}},$$

which implies $t_j^{1/2p_j} < 3t_j^{1/2}$. Hence

$$\sum_{j \notin J_1} j^{3/2}[\log(j+1)]^{\beta/2} 2^{j/2} \big[t_j\big]^{1/2p_j} \leq 3 \sum_{j=1}^{\infty} j^{3/2}[\log(j+1)]^{\beta/2} 2^{j/2} \big[t_j\big]^{1/2} \stackrel{\text{def}}{=} B_1. \tag{13}$$

Suppose that $\alpha > \delta > 2$. By rewriting and applying the Cauchy–Schwartz inequality we obtain with a suitable constant C_δ that

$$B_1 = 3 \sum_{j=1}^{\infty} \big[j^{3/2} j^{-\delta}\big] j^{\delta} \big[\log(j+1)\big]^{\beta/2} 2^{j/2} \big[t_j\big]^{1/2} \leq$$

$$3 \Big[\sum_{j=1}^{\infty} j^{3-2\delta}\Big]^{1/2} \Big[\sum_{j=1}^{\infty} j^{2\delta} \big[\log(j+1)\big]^{\beta} 2^j t_j\Big]^{1/2} =$$

$$C_\delta \Big[\sum_{j=1}^{\infty} j^{2\delta} \big[\log(j+1)\big]^{\beta} 2^j t_j\Big]^{1/2} \stackrel{\text{def}}{=} B_2.$$

There exists $C_{\delta,\alpha,\beta}$ such that for all $j = 1, 2, \ldots$

$$\big[\log(j+1)\big]^{\beta} \leq C_{\delta,\alpha,\beta} j^{2(\alpha-\delta)}.$$

Hence,

$$B_1 \leq B_2 \leq C_\delta C_{\delta,\alpha,\beta} \left(\int |f| (\log^+ |f|)^{2\alpha} d\mu\right)^{1/2}. \tag{14}$$

By (11-14) we have

$$\mu\{x : R^*(f)(x) > \lambda\} \leq \widehat{C} \frac{C_s + C_\delta C_{\delta,\alpha,\beta} (\int |f| (\log^+ |f|)^{2\alpha} d\mu)^{1/2}}{\lambda^{1/2}}$$

this implies (7) with a suitable C_α.

$\square$

Remark 1. Inequality (7) implies also that for the pair of nonnegative functions (f, g) in $(L(\log L)^{2\alpha}, L^1)$ we have

$$\lim_n \frac{f(T^n x) g(T^{2n} x)}{n} = 0. \tag{15}$$

Indeed, consider a sequence of bounded functions $0 \leq f_M \leq f$ converging monotone increasingly to $f \in L(\log L)^{2\alpha}$. Then we have

$$\lim_n \frac{f_M(T^n x) g(T^{2n} x)}{n} = 0. \tag{16}$$

Given $\varepsilon \in (0, 1)$ choose M so large that

$$I(M, \varepsilon, 1/2) \stackrel{\text{def}}{=} \left(\int \frac{2}{\varepsilon^2} |f - f_M| (\log^+ \frac{2}{\varepsilon^2} |f - f_M|)^{2\alpha} d\mu\right)^{1/2} < 1. \tag{17}$$

Then

$$\mu\{x : \limsup_{n\to\infty} \frac{f(T^n x)g(T^{2n}x)}{n} > \varepsilon\} \leq$$

$$\mu\{x : \limsup_{n\to\infty} \frac{(f-f_M)(T^n x)g(T^{2n}x)}{n} > \frac{\varepsilon}{2}\} + \mu\{x : \limsup_{n\to\infty} \frac{f_M(T^n x)g(T^{2n}x)}{n} > \frac{\varepsilon}{2}\} \leq$$

(by using (16))

$$\mu\{x : R^*((f-f_M), g)(x) > \frac{\varepsilon}{2}\} = \mu\{x : R^*(\frac{2}{\varepsilon^2}(f-f_M), g)(x) > \frac{1}{\varepsilon}\} \leq$$

(by using (7) and (17))

$$C_\alpha\sqrt{\varepsilon}(1 + I(M, \varepsilon, 1/2)) \leq 2C_\alpha\sqrt{\varepsilon}.$$

Since this holds for any $\varepsilon \in (0,1)$ we obtained (15).

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